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Generalized Fourier law

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Abstract—A generalized Fourier law is shown to follow logically from the principle of frame-indifference and the second law of thermodynamics. It can be used in the process of heat transfer in which a relative macroscopic motion is present between two sides exchanging heat. The classical Fourier law is recovered as a special case of it.

1. INTRODUCTION

THE CLASSICAL Fourier law is a fundamental law in heat transfer. It relates heat flux to the temperature gradient. As an experimental law, it is used to calculate the heat flux through a process in which no macroscopic relative motion is present between two sides which exchange heat. Two such processes are heat conduction in solids and convective heat transfer between a *viscous* fluid and a solid wall. In practical engineering processes, however, it is common that relative motion coexists with the heat transfer between two sides exchanging heat. A typical example is convective heat transfer in moving fluids. Even in solids, as well as the heat transfer the body usually undergoes deformation motion. Experience shows that such a relative motion would significantly affect heat transfer density or heat flux. Therefore, it is necessary to extend the classical Fourier law to the processes of heat transfer accompanied by the macroscopic relative motion between two sides exchanging heat.

The convective heat transfer study employs analytical, numerical or experimental methods to get velocity and temperature fields of the fluid. Then, the Fourier law is used to calculate the heat flux between the fluid and the wall. It is the limitation mentioned above of the classical Fourier law that precludes one from obtaining the heat flux distribution in the fluid even though the velocity and temperature distributions are in agreement. Such a heat flux distribution, however, plays an important role for proposing techniques of heat transfer enhancement or insulation, and is required in many fields. Shown in Fig. 1 is a typical pressure-driven vortex flow in a rotating curved square channel. The symmetry of the flow allows one to show only half the cross section of the channel. Heat transfer among the vortices is pre-required in order to study the evolution, stability of the vortices and transition to turbulent flow.

Joseph Fourier, an outstanding mathematician and physicist, proposed a proportional relation between heat flux and temperature gradient in 1807 based on experiences and investigations, which is now called the Fourier law [1, 2]. It has been confirmed by many experiments, i.e. it holds for many media in the usual temperature gradient range. However, much evidence shows that this proportional relation no longer holds if the temperature gradient is large.

As a typical example of Cauchy fluxes [3–5], the heat flux was shown to exist as a vector field whose scalar product with the normal to the surface results in the surface density of the flux. Cauchy [6] first established a result of this type in 1823 under an assumption that the density depends only on the normal. Such an assumption was shown to be essentially a consequence of the other assumptions on the flux by Noll in 1957 [7]. This inspired new research which led to important developments as shown in refs. [3, 4] and [8–13].

Believing that the classical Fourier law should be regarded as a limiting approximation, valid only for sufficiently homogeneous temperature fields, to some general nonlinear constitutive assumption for heat flux, Coleman and Mizel [14] assumed that the heat flux is a smooth, but nonlinear, function of the temperature and the first n spatial gradients of the temperature for the process of heat transfer in rigid bodies. By employing the method of Coleman and Noll [15], they introduced thermodynamics to the analysis. The consideration of thermodynamics and symmetry, taken together, yielded a complete fourth-order theory of heat conduction [14].

The classical Fourier law leads to a field equation for temperature which allows wave propagation at an infinite speed. This was observed by Cattaneo in 1948 [16]. Starting from Maxwell's idea [17] and from the paper by Cattaneo [16], an extensive amount of literature [18–28] has contributed to the elimination of the

NOMENCLATURE

$\mathbf{a}$	any vector
$\mathbf{A}$	any symmetric tensor
$\mathbf{b}$	any vector
$\mathbf{B}$	any symmetric tensor
$\mathbf{c}$	any vector
$\mathbf{D}$	velocity strain tensor
$\mathbf{e}_k$	eigenvector
$\mathbf{f}$	vector-valued function
$\mathbf{F}$	any invertible tensor or deformation gradient tensor
$\mathbf{f}_k$	eigenvector
J_k	invariant
$\mathbf{K}$	thermal conductivity tensor
$\mathbf{L}$	velocity gradient tensor
$\mathbf{q}$	heat-flux vector
$\mathbf{Q}$	rotation tensor
$\hat{\mathbf{Q}}$	rotation tensor
r	polar coordinate
$\mathbf{r}$	position vector
t	time
T.P.	thermophysical properties
$\mathbf{u}$	velocity of plate
$\mathbf{v}$	velocity
$\mathbf{W}$	skew part of tensor $\mathbf{L}$.

Greek symbols

α	coefficient
β	coefficient
γ	coefficient
η	scalar-valued function determined by energy equation
θ	temperature
μ_k	eigenvalue of symmetric tensor
τ	time instant
φ	polar coordinate
ϕ	scalar-valued function
ϕ_i	scalar-valued function
ψ	scalar-valued function
ψ_i	scalar-valued function
$\hat{\psi}_i$	scalar-valued function
ω	scalar-valued function determined by equation of motion
$\omega'(r, t)$	$\partial^2\omega/\partial t\,\partial r$
$\hat{\mathbf{\Omega}}$	skew tensor.

Subscripts

b	bottom plate (surface)
i	index, inner surface
j	index
k	index
o	outer surface
t	top plate (surface).

Superscripts

*	quantity observed by observer *
t	transpose of tensor
-1	inverse of tensor.

Other symbols

∇	gradient
$\forall$	for all.

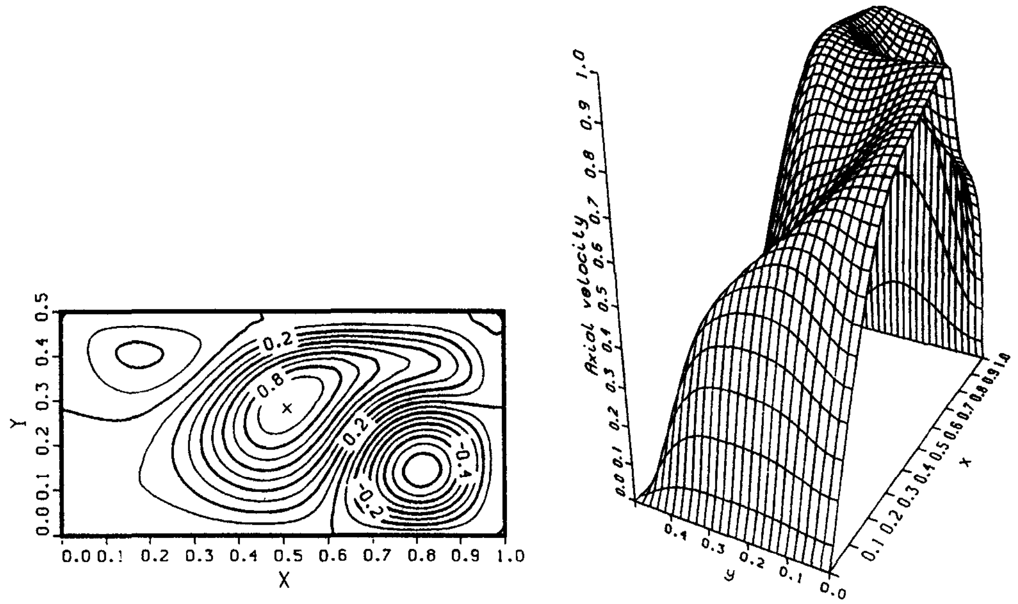
'paradox of instantaneous propagation of thermal disturbances'. The approach used is known as extended irreversible thermodynamics, which introduces time derivatives of the heat-flux vector, Cauchy stress tensor and its trace into the classical Fourier law by preserving the entropy principle. To do so, extended irreversible thermodynamics considers that the non-equilibrium entropy density not only depends on the two thermodynamic variables but also on the heat flux vector and the Cauchy stress tensor. The relaxation effects, then, were introduced by a simple modification of Gibbs relation. The Fourier law modified in this way established an implicit equation relating heat-flux vector, velocity and temperature.

Rational thermodynamics, a different approach to similar problems, derives the restrictions that the second law of thermodynamics (in the form of the Clausius–Duhem inequality due to Clausius, Duhem, and Truesdell and Toupin [29]) places on the heat flux vector [10, 14, 15, 30–35]. It was shown that heat-flux vector must satisfy the Fourier inequality [15, 32], which states that the projection of the heat-flux vector on the temperature-gradient vector is non-positive. An important consequence of the Fourier inequality is the non-existence of a piezo-caloric effect [15, 32]. The energy in the form of heat is, therefore, transferred from one body to another body (or from one part of a body to another part of the same body) only when the bodies are at different temperatures. The thermal conductivity tensor $\mathbf{K}$, defined by

$$\mathbf{q} = -\mathbf{K}\nabla\theta$$

was assumed in general to be a function of deformation gradient $\mathbf{F}$ and temperature θ [32]. By imposing the principle of frame-indifference, $\mathbf{K}$ was shown to be a symmetric tensor by Wang [32]. This condition is one of the so-called Onsager relations [32]. The detailed form of $\mathbf{K}$ as a function of $\mathbf{F}$, however, still remains to be established.

The motivation of the present work comes from the desire to establish the relation between heat flux and temperature gradient in the heat transfer process with macroscopic relative motion between two sides exchanging heat. In this paper, the result is termed as the generalized Fourier law, which is shown to follow logically from the second law of thermodynamics and



(a) secondary flow (b) main flow

FIG. 1. A typical vortex flow in a rotating curved square channel.

the principle of frame-indifference, and the classical Fourier law is recovered as a special case of it. The thermal conductivity is found to be a function of temperature, temperature gradient and thermophysical properties of the medium in general.

2. VELOCITY VS HEAT FLUX

Lemma 1

Heat flux is independent of velocity.

Proof. Experiences show that heat flux $\mathbf{q}$ is dependent on the temperature θ, thermophysical properties of the medium, temperature gradient $\nabla\theta$ and velocity gradient tensor $\mathbf{L}$. Suppose that it is also dependent on the velocity of the medium $\mathbf{v}$ itself, then

$$\mathbf{q} = \mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}, \mathbf{v}) \tag{1}$$

where $\mathbf{f}$ is a vector-valued function. T.P. represents the thermophysical properties of the medium.

For another observer *, the principle of frame-indifference gives

$$\mathbf{q}^* = \mathbf{f}(\theta^*, (\text{T.P.})^*, (\nabla\theta^*)^*, \mathbf{L}^*, \mathbf{v}^*) \tag{2}$$

in which superscript * represents the quantities observed by observer *.

From the principle of observer transformations [36],

$$\left.\begin{array}{ll} \theta^* = \theta & (\text{T.P.})^* = \text{T.P.} \\ \mathbf{q}^* = \mathbf{Q}\mathbf{q} & (\nabla\theta^*)^* = \mathbf{Q}(t)\nabla\theta \\ \mathbf{L}^* = \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} & \mathbf{r}^* = \mathbf{Q}\mathbf{r} + \mathbf{c}(t) \\ \mathbf{v}^* = \mathrm{d}r^*/\mathrm{d}t = \dot{\mathbf{Q}}\mathbf{r} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}(t) & \end{array}\right\} \tag{3}$$

where $\mathbf{Q}$ is an arbitrary rotation tensor, $\mathbf{r}$ a position vector of material point, and $\mathbf{c}(t)$ is an arbitrary vector-valued function of time t.

On substitution of expressions (1) and (3) into expression (2), one has

$$\mathbf{f}(\theta, \text{T.P.}, \mathbf{Q}\nabla\theta, \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}, \dot{\mathbf{Q}}\mathbf{r} + \mathbf{Q}\mathbf{v} + \dot{\mathbf{c}}(t)) = \mathbf{Q}\mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}, \mathbf{v}) \quad \forall\mathbf{Q} \text{ and } \mathbf{c}(t). \tag{4}$$

Since expression (4) is true for all $\mathbf{Q}$, it must be true for $\mathbf{Q} = \mathbf{1}$. Take $\mathbf{Q} = \mathbf{1}$, then $\dot{\mathbf{Q}} = \mathbf{0}$. Equation (4) gives

$$\mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}, \mathbf{v} + \dot{\mathbf{c}}) = \mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}, \mathbf{v}) \quad \forall\dot{\mathbf{c}}(t). \tag{5}$$

This implies that $\mathbf{f}$ is independent of velocity. Expressions (1) and (4) are, then, simplified as

$$\mathbf{q} = \mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}) \tag{6}$$

$$\mathbf{f}(\theta, \text{T.P.}, \mathbf{Q}\nabla\theta, \mathbf{Q}\mathbf{L}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}) \quad \forall\mathbf{Q}. \tag{7}$$

3. VELOCITY GRADIENT TENSOR VS HEAT FLUX

Since $\mathbf{L}$ can be written as the sum of a symmetric tensor $\mathbf{D}$ (velocity strain tensor) and a skew tensor $\mathbf{W}$, then expression (7) may be rewritten as

$$\mathbf{f}(\theta, \text{T.P.}, \mathbf{Q}\nabla\theta, \mathbf{Q}\mathbf{D}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}) = \mathbf{Q}\mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{L}) \quad \forall\mathbf{Q}. \tag{8}$$

It can be easily shown that $\mathbf{D}$ and $\mathbf{W}$ are unique.

Lemma 2

For any fixed time τ and all time t

$$\mathbf{Q}(t) = \exp[\hat{\mathbf{\Omega}}(t-\tau)] = \sum_{n=0}^{\infty} \frac{(t-\tau)^n}{n!}\hat{\mathbf{\Omega}}^n$$

is a rotation tensor provided that $\hat{\mathbf{\Omega}}$ is a fixed skew tensor.

Proof. From the theory of series, one may conclude that the series

$$\sum_{n=0}^{\infty} \frac{(t-\tau)^n}{n!}\hat{\mathbf{\Omega}}^n$$

is absolutely convergent to $\exp[\hat{\mathbf{\Omega}}(t-\tau)]$.

Let

$$\mathbf{Q}(t) = \exp[\hat{\mathbf{\Omega}}(t-\tau)] = \sum_{n=0}^{\infty} \frac{(t-\tau)^n}{n!}\hat{\mathbf{\Omega}}^n.$$

Then

$$\mathbf{Q}(\tau) = \mathbf{1}$$

$$\dot{\mathbf{Q}}(t) = \frac{\mathrm{d}\mathbf{Q}(t)}{\mathrm{d}t} = \hat{\mathbf{\Omega}} + (t-\tau)\hat{\mathbf{\Omega}}^2 + \tfrac{1}{2}(t-\tau)^2\hat{\mathbf{\Omega}}^3 + \ldots$$

$$= \hat{\mathbf{\Omega}}[1 + (t-\tau)\hat{\mathbf{\Omega}} + \tfrac{1}{2}(t-\tau)^2\hat{\mathbf{\Omega}}^2 + \tfrac{1}{6}(t-\tau)^3\hat{\mathbf{\Omega}}^3 + \ldots] = \hat{\mathbf{\Omega}}\mathbf{Q}(t)$$

and

$$(\mathbf{Q}^{\mathrm{T}}\mathbf{Q})^{\cdot} = \mathbf{Q}^{\mathrm{T}}(\hat{\mathbf{\Omega}} + \hat{\mathbf{\Omega}}^{\mathrm{T}})\mathbf{Q} = \mathbf{0}$$

if $\hat{\mathbf{\Omega}}$ is a skew tensor. Therefore

$$\mathbf{Q}^{\mathrm{T}}(t)\mathbf{Q}(t) = \mathbf{Q}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau) = \mathbf{1} \quad \forall t. \tag{9}$$

Since $(\det \mathbf{F})^{\cdot} = \det \mathbf{F} \operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1})$ for all invertible $\mathbf{F}$ [37], then

$$(\det \mathbf{Q})^{\cdot} = \det \mathbf{Q} \operatorname{tr}(\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}) = \det \mathbf{Q} \operatorname{tr}(\hat{\mathbf{\Omega}}\mathbf{Q}\mathbf{Q}^{\mathrm{T}}) = \det \mathbf{Q} \operatorname{tr}(\hat{\mathbf{\Omega}}) = 0.$$

Therefore,

$$\det \mathbf{Q}(t) = \det \mathbf{Q}(\tau) = \det \mathbf{1} = 1 \quad \forall t. \tag{10}$$

Expressions (9) and (10) are the mathematical form of Lemma 2.

Lemma 3

For rotation tensor $\mathbf{Q}(t) = \exp[\hat{\mathbf{\Omega}}(t-\tau)]\hat{\mathbf{Q}}$, one can pick $\mathbf{Q}(\tau)$ and $\mathbf{Q}(\tau)\mathbf{Q}^{\mathrm{T}}(\tau)$ to be arbitrary, independent rotation and skew tensors respectively at any instant τ, where $\hat{\mathbf{Q}}$ is any fixed rotation tensor, and $\hat{\mathbf{\Omega}}$ is any fixed skew tensor.

Proof. Since both $\hat{\mathbf{Q}}$ and $\exp[\hat{\mathbf{\Omega}}(t-\tau)]$ are rotation tensors (Lemma 2), then $\mathbf{Q}(t) = \exp[\hat{\mathbf{\Omega}}(t-\tau)]\hat{\mathbf{Q}}$ is also a rotation tensor for all time t, and

$$\mathbf{Q}(\tau) = \hat{\mathbf{Q}} \tag{11}$$

$$\dot{\mathbf{Q}}(\tau)\mathbf{Q}^{\mathrm{T}}(\tau) = \hat{\mathbf{\Omega}}\mathbf{Q}(\tau)\mathbf{Q}^{\mathrm{T}}(\tau) = \hat{\mathbf{\Omega}}. \tag{12}$$

They are clearly independent rotation and skew tensors if $\hat{\mathbf{Q}}$ and $\hat{\mathbf{\Omega}}$ are arbitrary fixed rotation and skew tensors respectively.

Lemma 4

L affects heat flux only through velocity strain tensor **D**.

Proof. To prove this, choose $\mathbf{Q}(t)$ defined in Lemma 3 as the rotation tensor in expression (8) while, for any instant τ, $-\mathbf{QWQ}^{\mathrm{T}}|_\tau$ is used as the skew tensor $\hat{\mathbf{\Omega}}$, i.e. $\hat{\mathbf{\Omega}} = -\mathbf{QWQ}^{\mathrm{T}}|_\tau$ (such an $\hat{\mathbf{\Omega}}$ is a skew tensor since $\hat{\mathbf{\Omega}}^{\mathrm{T}} = -\mathbf{QW}^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}|_\tau = \mathbf{QWQ}^{\mathrm{T}}|_\tau = -\hat{\mathbf{\Omega}}$). Then, at time $t = \tau$, expression (8) gives

$$\mathbf{f}(\theta, \mathrm{T.P.}, \hat{\mathbf{Q}}\nabla\theta, \hat{\mathbf{Q}}\mathbf{D}\hat{\mathbf{Q}}^{\mathrm{T}}) = \hat{\mathbf{Q}}\mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{L}) \quad \forall\hat{\mathbf{Q}}. \tag{13}$$

Since this holds for all rotation tensors $\hat{\mathbf{Q}}$, it must hold for $\hat{\mathbf{Q}} = \mathbf{1}$. Let $\hat{\mathbf{Q}} = \mathbf{1}$, then expression (13) gives

$$\mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{L}) = \mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{D}) \tag{14}$$

or

$$\mathbf{q} = \mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{D}). \tag{15}$$

4. TEMPERATURE GRADIENT VS HEAT FLUX

Expression (15) and the principle of frame-indifference together give

$$\mathbf{q}^* = \mathbf{f}(\theta^*, (\mathrm{T.P.})^*, (\nabla\theta^*)^*, \mathbf{D}^*). \tag{16}$$

On substitution of expressions (3), (15) and $\mathbf{D}^* = \mathbf{QDQ}^{\mathrm{T}}$ [36] into expression (16), one gets

$$\mathbf{f}(\theta, \mathrm{T.P.}, \mathbf{Q}\nabla\theta, \mathbf{QDQ}^{\mathrm{T}}) = \mathbf{Q}\mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{D}) \quad \forall\mathbf{Q}. \tag{17}$$

Also the second law of thermodynamics requires

$$\mathbf{f}(\theta, \mathrm{T.P.}, -\nabla\theta, \mathbf{D}) = -\mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{D}). \tag{18}$$

Since the velocity strain tensor **D** is a real, symmetric tensor, it has three real eigenvalues. The three eigenvalues can be distinct, identical, or two of them can be identical. In this paper, a necessary condition of expressions (17) and (18) is to be found only under the condition that the three eigenvalues are distinct. Similar results may be obtained for the other two cases.

Lemma 5

$\nabla\theta$, $\mathbf{D}\nabla\theta$, $\mathbf{D}^2\nabla\theta$ are linearly independent if three eigenvalues of **D** are distinct.

Proof. Let μ_k and $\mathbf{f}_k$ ($k = 1, 2, 3$) be the eigenvalues and eigenvectors of **D**. Then **D** may be represented, in its spectral form, as

$$\mathbf{D} = \sum_{k=1}^{3} \mu_k \mathbf{f}_k \otimes \mathbf{f}_k. \tag{19}$$

The linear independence of $\mathbf{f}_k$ ($k = 1, 2, 3$) allows one to write $\nabla\theta$ as

$$\nabla\theta = (\nabla\theta)_j \mathbf{f}_j \tag{20}$$

in which $(\nabla\theta)_j = \nabla\theta \cdot \mathbf{f}_j$.

Suppose that $\nabla\theta$, $\mathbf{D}\nabla\theta$ and $\mathbf{D}^2\nabla\theta$ are linearly dependent for all $\mathbf{D}$ and $\nabla\theta$; there are α, β and γ which are not all zero, such that

$$\alpha\nabla\theta + \beta\mathbf{D}\nabla\theta + \gamma\mathbf{D}^2\nabla\theta = \mathbf{0}. \tag{21}$$

Substituting expressions (19) and (20) into expression (21) gives

$$\sum_{k=1}^{3} (\alpha + \beta\mu_k + \gamma\mu_k^2)(\nabla\theta)_k \mathbf{f}_k = 0. \tag{22}$$

This implies

$$(\alpha + \beta\mu_k + \gamma\mu_k^2)(\nabla\theta)_k = 0 \quad (k = 1, 2, 3) \tag{23}$$

since $\mathbf{f}_k$ (k = 1, 2, 3) are linearly independent.

For arbitrary $\nabla\theta$, $(\nabla\theta)_k$ need not be zero, so

$$\alpha + \beta\mu_k + \gamma\mu_k^2 = 0 \quad (k = 1, 2, 3) \tag{24}$$

and this requires $\alpha = \beta = \gamma = 0$ for distinct μ_k. This is contrary to the hypothesis. Lemma 5, then, has been proved.

Applying Lemma 5 to the heat flux vector, one has

$$\mathbf{f}(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) = \phi_0(\theta, \text{T.P.}, \nabla\theta, \mathbf{D})\nabla\theta + \phi_1(\theta, \text{T.P.}, \nabla\theta, \mathbf{D})\mathbf{D}\nabla\theta + \phi_2(\theta, \text{T.P.}, \nabla\theta, \mathbf{D})\mathbf{D}^2\nabla\theta \tag{25}$$

and

$$\mathbf{f}(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D}) = -\phi_0(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D})\nabla\theta - \phi_1(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D})\mathbf{D}\nabla\theta - \phi_2(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D})\mathbf{D}^2\nabla\theta. \tag{26}$$

Substituting (25) and (26) into (18) gives

$$[\phi_0(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) - \phi_0(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D})]\nabla\theta + [\phi_1(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) - \phi_1(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D})]\mathbf{D}\nabla\theta + [\phi_2(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) - \phi_2(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D})]\mathbf{D}^2\nabla\theta = 0. \tag{27}$$

This implies

$$\phi_i(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) = \phi_i(\theta, \text{T.P.}, -\nabla\theta, \mathbf{D}) \quad (i = 0, 1, 2) \tag{28}$$

since $\nabla\theta$, $\mathbf{D}\nabla\theta$ and $\mathbf{D}^2\nabla\theta$ are linearly independent.

To satisfy this requirement, take

$$\phi_i(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) = \psi_i(\theta, \text{T.P.}, \nabla\theta\otimes\nabla\theta, \mathbf{D}) \quad (i = 0, 1, 2). \tag{29}$$

Then expressions (25) and (29) give

$$\mathbf{f}(\theta, \text{T.P.}, \mathbf{Q}\nabla\theta, \mathbf{QDQ}^{\mathrm{T}}) = \mathbf{Q}(\hat{\psi}_0\nabla\theta + \hat{\psi}_1\mathbf{D}\nabla\theta + \hat{\psi}_2\mathbf{D}^2\nabla\theta) \tag{30}$$

in which

$$\hat{\psi}_i = \psi_i(\theta, \text{T.P.}, \mathbf{Q}\nabla\theta\otimes\mathbf{Q}\nabla\theta, \mathbf{QDQ}^{\mathrm{T}})$$

and

$$\mathbf{Qf}(\theta, \text{T.P.}, \nabla\theta, \mathbf{D}) = \mathbf{Q}(\psi_0\nabla\theta + \psi_1\mathbf{D}\nabla\theta + \psi_2\mathbf{D}^2\nabla\theta). \tag{31}$$

Substituting expressions (30) and (31) into (17) gives

$$(\hat{\psi}_0 - \psi_0)\nabla\theta + (\hat{\psi}_1 - \psi_1)\mathbf{D}\nabla\theta + (\hat{\psi}_2 - \psi_2)\mathbf{D}^2\nabla\theta = \mathbf{0}. \tag{32}$$

This implies, by Lemma 5, that

$$\psi_i(\theta, \text{T.P.}, \mathbf{Q}\nabla\theta\otimes\mathbf{Q}\nabla\theta, \mathbf{QDQ}^{\mathrm{T}}) = \psi_i(\theta, \text{T.P.}, \nabla\theta\otimes\nabla\theta, \mathbf{D}) \quad \forall\mathbf{Q}. \tag{33}$$

Lemma 6

Suppose

$$\psi(\theta, \text{T.P.}, \mathbf{Qb}\otimes\mathbf{Qb}, \mathbf{QBQ}^{\mathrm{T}}) = \psi(\theta, \text{T.P.}, \mathbf{b}\otimes\mathbf{b}, \mathbf{B}) \quad \forall\mathbf{b}\text{ and }\mathbf{B}$$

then

$$\psi(\theta, \text{T.P.}, \mathbf{a}\otimes\mathbf{a}, \mathbf{A}) = \psi(\theta, \text{T.P.}, \mathbf{b}\otimes\mathbf{b}, \mathbf{B})$$

whenever $J_k(\mathbf{a}, \mathbf{A}) = J_k(\mathbf{b}, \mathbf{B})$ ($k = 1, 2, \ldots, 6$). Where

$$J_1(\mathbf{a}, \mathbf{A}) = \operatorname{tr}\mathbf{A} \qquad J_2(\mathbf{a}, \mathbf{A}) = \tfrac{1}{2}[(\operatorname{tr}\mathbf{A})^2 - \operatorname{tr}(\mathbf{A}^2)]$$

$$J_3(\mathbf{a}, \mathbf{A}) = \det\mathbf{A} \qquad J_4(\mathbf{a}, \mathbf{A}) = \mathbf{a}\cdot\mathbf{Aa}$$

$$J_5(\mathbf{a}, \mathbf{A}) = \mathbf{a}\cdot\mathbf{A}^2\mathbf{a} \qquad J_6(\mathbf{a}, \mathbf{A}) = |\mathbf{a}|$$

and $\mathbf{a}$ and $\mathbf{b}$ are two arbitrary vectors, $\mathbf{A}$ and $\mathbf{B}$ are two arbitrary symmetric tensors.

Proof. Since $J_k(\mathbf{a}, \mathbf{A}) = J_k(\mathbf{b}, \mathbf{B})$ (k = 1, 2, 3), tensors $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues. Let μ_k be their eigenvalues, then $\mathbf{A}$ and $\mathbf{B}$ may be written as

$$\mathbf{A} = \sum_{k=1}^{3} \mu_k\mathbf{e}_k\otimes\mathbf{e}_k \quad \mathbf{B} = \sum_{k=1}^{3} \mu_k\mathbf{f}_k\otimes\mathbf{f}_k$$

where $\mathbf{e}_k$ and $\mathbf{f}_k$ (k = 1, 2, 3) are eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ respectively.

Define

$$\mathbf{Q} = \mathbf{e}_k\otimes\mathbf{f}_k$$

then $\mathbf{Q}$ is a rotation tensor, and

$$\mathbf{e}_i = \mathbf{Qf}_i \quad \mathbf{A} = \mathbf{QBQ}^{\mathrm{T}} \quad \mathbf{A}^2 = \mathbf{QB}^2\mathbf{Q}^{\mathrm{T}} \quad (i = 1, 2, 3). \tag{34}$$

Then, applying $J_k(\mathbf{a}, \mathbf{A}) = J_k(\mathbf{b}, \mathbf{B})$ (k = 4, 5, 6), one gets

$$\left.\begin{aligned}\sum_{k=1}^{3}(\mathbf{b}\cdot\mathbf{f}_k)^2 &= \sum_{k=1}^{3}(\mathbf{Q}^{\mathrm{T}}\mathbf{a}\cdot\mathbf{f}_k)^2\\ \sum_{k=1}^{3}\mu_k(\mathbf{b}\cdot\mathbf{f}_k)^2 &= \sum_{k=1}^{3}\mu_k(\mathbf{Q}^{\mathrm{T}}\mathbf{a}\cdot\mathbf{f}_k)^2\\ \sum_{k=1}^{3}\mu_k^2(\mathbf{b}\cdot\mathbf{f}_k)^2 &= \sum_{k=1}^{3}\mu_k^2(\mathbf{Q}^{\mathrm{T}}\mathbf{a}\cdot\mathbf{f}_k)^2.\end{aligned}\right\}. \tag{35}$$

This implies

$$(\mathbf{Q}^{\mathrm{T}}\mathbf{a}\mp\mathbf{b})\cdot\mathbf{f}_k = 0 \quad (k=1,2,3) \tag{36}$$

if μ_k $(k = 1, 2, 3)$ are distinct.

Note that, if $\mathbf{f}_k$ $(k = 1, 2, 3)$ are linearly independent, then

$$\mathbf{a} = \pm\mathbf{Q}\mathbf{b} \quad \mathbf{a}\otimes\mathbf{a} = \mathbf{Q}\mathbf{b}\otimes\mathbf{Q}\mathbf{b}.$$

By hypothesis,

$$\begin{aligned}\psi(\theta, \mathrm{T.P.}, \mathbf{b}\otimes\mathbf{b}, \mathbf{B}) &= \psi(\theta, \mathrm{T.P.}, \mathbf{Q}\mathbf{b}\otimes\mathbf{Q}\mathbf{b}, \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}})\\ &= \psi(\theta, \mathrm{T.P.}, \mathbf{a}\otimes\mathbf{a}, \mathbf{A})\end{aligned}$$

in which $\mathbf{A} = \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}$ (expression (34)) and $\mathbf{a}\otimes\mathbf{a} = \mathbf{Q}\mathbf{b}\otimes\mathbf{Q}\mathbf{b}$ are used. That is,

$$\psi(\theta, \mathrm{T.P.}, \mathbf{b}\otimes\mathbf{b}, \mathbf{B}) = \psi[\theta, \mathrm{T.P.}, J_k(\mathbf{b}, \mathbf{B})] \quad (k = 1, 2, \ldots, 6) \tag{37}$$

if

$$\psi(\theta, \mathrm{T.P.}, \mathbf{Q}\mathbf{b}\otimes\mathbf{Q}\mathbf{b}, \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}) = \psi(\theta, \mathrm{T.P.}, \mathbf{b}\otimes\mathbf{b}, \mathbf{B}) \quad \forall\mathbf{b} \text{ and } \mathbf{B}. \tag{38}$$

The converse is also true since $J_k(\mathbf{Q}\mathbf{b}, \mathbf{Q}\mathbf{B}\mathbf{Q}^{\mathrm{T}}) = J_k(\mathbf{b}, \mathbf{B})$ $(k = 1, 2, \ldots, 6)$.

Lemma 7

The principle of frame-indifference and the second law of thermodynamics require the heat flux to be

$$\mathbf{q} = \mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{D}) = (\phi_0\mathbf{1}+\phi_1\mathbf{D}+\phi_2\mathbf{D}^2)\nabla\theta$$

where

$$\phi_i = \phi_i[\theta, \mathrm{T.P.}, J_k(\nabla\theta, \mathbf{D})]$$
$$(i = 0, 1, 2 \quad k = 1, 2, \ldots, 6)$$

Proof. Applying Lemma 6 to expression (33), one has

$$\psi_i(\theta, \mathrm{T.P.}, \nabla\theta\otimes\nabla\theta, \mathbf{D}) = \psi_i[\theta, \mathrm{T.P.}, J_k(\nabla\theta, \mathbf{D})]. \tag{39}$$

Substituting expressions (39) and (29) into (25) gives

$$\mathbf{q} = \mathbf{f}(\theta, \mathrm{T.P.}, \nabla\theta, \mathbf{D}) = (\phi_0\mathbf{1}+\phi_1\mathbf{D}+\phi_2\mathbf{D}^2)\nabla\theta \tag{40}$$

where

$$\phi_i = \phi_i[\theta, \mathrm{T.P.}, J_k(\nabla\theta, \mathbf{D})]$$
$$(i = 0, 1, 2 \quad k = 1, 2, \ldots, 6).$$

This is the generalized Fourier law. It is obvious that $\mathbf{q} = 0$ if $\nabla\theta = 0$. There is no piezo-caloric effect. This is in agreement with refs. [14, 15, 32].

If the three eigenvalues of $\mathbf{D}$ are not distinct, one can still obtain expression (40) with $\phi_1 = \phi_2 = 0$ (for the case of three identical eigenvalues) or $\phi_2 = 0$ (for the case of two identical eigenvalues) by a similar method. Therefore, expression (40) is valid for all cases.

5. DISCUSSION

Rewrite the generalized Fourier law (40) as

$$\mathbf{q} = -\mathbf{K}\nabla\theta$$

then

$$\mathbf{K} = -(\phi_0\mathbf{1}+\phi_1\mathbf{D}+\phi_2\mathbf{D}^2). \tag{41}$$

Symmetry of velocity strain tensor $\mathbf{D}$, then, ensures that thermal conductivity tensor $\mathbf{K}$ is symmetric. This is in agreement with ref. [32].

Since heat flux $\mathbf{q}$ must satisfy Fourier inequality [14, 15, 32], i.e.

$$\mathbf{q}\cdot\nabla\theta \leqslant 0 \quad \forall\nabla\theta.$$

then

$$\nabla\theta\cdot\mathbf{K}\nabla\theta \geqslant 0 \quad \forall\nabla\theta \tag{42}$$

which implies that $\mathbf{K}$ is positive semi-definite. And since $\mathbf{K}$ is, in practice, an invertible tensor, it is positive definite. The same conclusion may be obtained by noting that the inequality in (42) is for irreversible processes and heat transfer is an irreversible process.

If the medium is at rest, has uniform motion or is in rigid body rotation, then

$$\mathbf{D} = \mathbf{0} \quad J_k = 0 \quad (k = 1, 2, \ldots, 5)$$

and the generalized Fourier law reduces to

$$\mathbf{q} = \phi_0(\theta, \mathrm{T.P.}, \nabla\theta)\nabla\theta.$$

This is the clasical Fourier law. Here, ϕ_0 is the thermal conductivity of the medium. It is, in general, a function of temperature, temperature gradient and the thermophysical properties of the medium.

The generalized Fourier law may be simplified for some special motions such as one-dimensional flow or axis-symmetrical flow of incompressible fluids. Two examples are heat transfers in processes of simple shear flow and axial shear motion.

For simple shear flow between two infinite parallel plates (Fig. 2), it is easy to show that

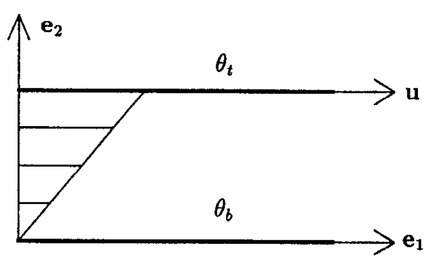


FIG. 2. Simple shear flow between two infinite parallel plates.

$$\mathbf{D} = \frac{\dot{\omega}(t)}{2}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$$

where $\omega(t)$ can be determined by solving the equation of motion. Temperature is governed by the energy equation which contains ϕ_0, ϕ_1 and ϕ_2. ϕ_i $(i = 0, 1, 2)$ is material-dependent and is to be determined through experiments. This prevents one from solving the energy equation to obtain temperature distribution at the present stage. For the case of constant temperatures of top and bottom plates, however, the temperature gradient in the fluid is expected to be parallel to $\mathbf{e}_2$ as the first-order approximation, i.e.

$$\nabla\theta = \eta(\mathbf{r}, t)\mathbf{e}_2$$

in which $\eta(\mathbf{r}, t)$ is to be determined by solving the energy equation, and the generalized Fourier law reduces to

$$\mathbf{q} = \eta(\mathbf{r}, t)\left[\frac{\dot{\omega}(t)}{2}\phi_1\mathbf{e}_1 + \left(\phi_0 + \frac{\dot{\omega}^2(t)}{4}\phi_2\right)\mathbf{e}_2\right]$$

with

$$J_1 = J_3 = J_4 = 0$$

$$J_2 = -\frac{\dot{\omega}^2(t)}{4}$$

$$J_5 = \frac{\dot{\omega}^2(t)\eta^2}{4}$$

and

$$J_6 = |\eta|$$

which states that $\mathbf{q}$ is not parallel to the temperature gradient in general.

Consider an axial shear motion of a circular cylindrical tube (Fig. 3); it is easy to show that

$$\mathbf{D} = \tfrac{1}{2}\dot{\omega}'(r, t)\,(\mathbf{e}_3 \otimes \mathbf{e}_r + \mathbf{e}_r \otimes \mathbf{e}_3).$$

This displaces the circle r = constant by the amount $\omega(r, t)$, which can be determined by the equation of motion, along the axis of the cylinder. Suppose that inner and outer surfaces remain at the same constant temperature. As the first-order approximation, one may expect that the temperature gradient is parallel to the axis of the cylinder, i.e.

$$\nabla\theta = \eta(\mathbf{r}, t)\mathbf{e}_3$$

when top and bottom surfaces remain at different constant temperatures. Here $\eta(\mathbf{r}, t)$ is to be determined by the energy equation. The generalized Fourier law, then, reduces to

$$\mathbf{q} = \eta(\mathbf{r}, t)\left[\frac{\dot{\omega}'(r, t)}{2}\phi_1\mathbf{e}_r + \left(\phi_0 + \frac{(\dot{\omega}'(r, t))^2}{4}\phi_2\right)\mathbf{e}_3\right]$$

which shows that the heat flux has an $\mathbf{e}_r$ component in general, although the temperature gradient is along the axis $\mathbf{e}_3$. For this case,

$$J_1 = J_3 = J_4 = 0$$

$$J_2 = -\left(\frac{\dot{\omega}'(r, t)}{2}\right)^2$$

$$J_5 = \left(\frac{\dot{\omega}'(r, t)}{2}\eta\right)^2$$

and

$$J_6 = |\eta|.$$

A striking feature from these two examples is that heat flux is not parallel to the temperature gradient, in general, for either of the two cases, although the temperature gradient and the velocity are perpendicular to each other for the case of simple shear flow, but parallel to each other for the case of axial shear motion.

Like thermal conductivity, the detailed expressions of ϕ_0, ϕ_1 and ϕ_2 are material-dependent and need to be determined through experiments. Once they are determined, the generalized Fourier law can serve as a tool for looking for techniques of heat transfer

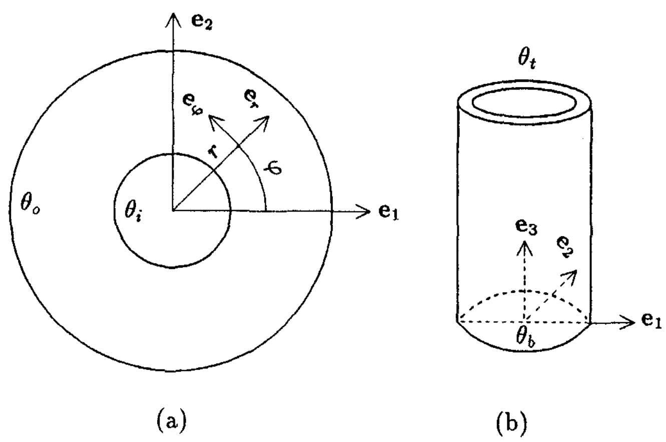


FIG. 3. Axial shear motion of a circular cylindrical tube.

enhancement or insulation. Furthermore, it also can be used to find new media satisfying certain requirements of heat transfer characteristics.

REFERENCES

1 K. C. Cheng, Historical development of convective heat transfer from Newton to Eckert (1700–1960). In *Transport Phenomena in Thermal Control* (Edited by G. J. Hwang), pp. 3–35. Hemisphere, New York (1989).
2. I. Grattan-Guinness, *Joseph Fourier* 1768–1830 (1st Edn), pp. 36–130. MIT Press, Cambridge, MA (1972).
3. M. E. Gurtin and L. C. Martins, Cauchy's theorem in classical physics, *Arch. Rational Mech. Anal.* **60**, 305–324 (1976).
4. W. P. Ziemer, Cauchy flux and sets of finite perimeter, *Arch. Rational Mech. Anal.* **84**, 189–201 (1983).
5. L. C. Martins, On Cauchy's theorem in classical physics: some counterexamples, *Arch. Rational Mech. Anal.* **60**, 325–328 (1976).
6. A. L. Cauchy, Recherches sur l'équilibre et le mouvement intérieur des corps solides ou fluides, élastiques ou non-élastiques, *Bull. Soc. Philomath.* 9–13 (1923).
7. W. Noll, The foundations of classical mechanics in the light of recent advances in continuum mechanics. In *The Axiomatic Methods, with Special Reference to Geometry and Physics*, pp. 266–281. North-Holland, Amsterdam (1959).
8. W. Noll, Lectures on the foundations of continuum mechanics, *Arch. Rational Mech. Anal.* **52**, 62–92 (1973).
9. W. Noll, The foundations of mechanics. In *Nonlinear Continuum Theories*, pp. 159–200. C.I.M.E. Lectures, Roma (1966).
10. M. E. Gurtin and W.O. Williams, An axiomatic foundation for continuum thermodynamics, *Arch. Rational Mech. Anal.* **26**, 83–117 (1967).
11. M. E. Gurtin, V. J. Mizel and W. O. Williams, A note on Cauchy's stress theorem, *J. Math. Anal. Appl.* **22**, 398–401 (1968).
12. M. Šilhavý, The existence of the flux vector and the divergence theorem for general Cauchy fluxes, *Arch. Rational Mech. Anal.* **90**, 195–212 (1985).
13. M. Šilhavý, Cauchy's stress theorem and tensor fields with divergences in L^p, *Arch. Rational Mech. Anal.* **116**, 223–255 (1991).
14. B. D. Coleman and V. J. Mizel, Thermodynamics and departures from Fourier's law of heat conduction, *Arch. Rational Mech. Anal.* **13**, 245–261 (1963).
15. B. D. Coleman and W. Noll, The thermodynamics of elastic materials with heat conduction and viscosity, *Arch. Rational Mech. Anal.* **13**, 167–178 (1963).
16. C. Cattaneo, Sulla conduzione del calore, *Atti. Sem. Mat. Fis. Modena* **3**, 83–101 (1948).
17. C. Truesdell and R. G. Muncaster, *Fundamentals of Maxwell's Kinetic Theory of a Simple Monatomic Gas* (1st Edn), pp. 187–196, 261–291. Academic Press, New York (1980).
18. M. Carassi and A. Morro, A modified Navier–Stokes equation, and its consequences on sound dispersion, *Nuovo Cimento* **9B**, 323–343 (1972).
19. W. Israel, Non-stationary irreversible thermodynamics: a casual relativistic theory, *Ann. Phys.* **100**, 310–331 (1976).
20. W. Israel and J. M. Stewart, Transient relativistic thermodynamics and kinetic theory, *Ann. Phys.* **118**, 341–372 (1979).
21. M. Kranyš, Relativistic hydrodynamics with irreversible thermodynamics without the paradox of infinite velocity of heat conduction. *Nuovo Cimento* **48B**, 51–70 (1966).
22. M. Kranyš, Kinetic derivation of nonstationary general relativistic thermodynamics, *Nuovo Cimento* **8B**, 417–441 (1972).
23. G. Lebon, Derivation of generalized Fourier and Stokes–Newton equations based on the thermodynamics of irreversible processes, *Acad. Roy. Soc. Belgique* **64**, 456–472 (1978).
24. G. Lebon, D. Jou and V. J. Casas, An extension of the local equilibrium hypothesis, *J. Phys.* **13A**, 275–290 (1980).
25. G. Lebon, D. Jou and V. J. Casas, A dynamical interpretation of second order constitutive equations of hydrodynamics, *J. Non-Equilib. Thermodyn.* **4**, 349–362 (1979).
26. G. Lebon and J. M. Rubi, A generalized theory of thermoviscous fluids, *J. Non-Equilib. Thermodyn.* **5**, 285–300 (1980).
27. I. Müller, Zum Paradoxen der Wärmeleitungstheorie, *Zeitschrift für Physik* **198**, 329–344 (1967).
28. T. Ruggeri, Symmetric-hyperbolic system of conservative equations for a viscous heat conducting fluid, *Acta Mechanica* **47**, 167–183 (1983).
29. C. Truesdell and R. A. Toupin, The classical field theories. In *Encyclopedia of Physics* (Edited by S. Flügge), Vol. III/1, pp. 226–793. Springer-Verlag, Berlin–Göttingen–Heidelberg (1960).
30. I. Müller, The coldness, a universal function in thermoelastic bodies, *Arch. Rational Mech. Anal.* **40**, 319–332 (1971).
31. M. E. Gurtin, On the thermodynamics of materials with memory, *Arch. Rational Mech. Anal.* **28**, 40–50 (1968).
32. C. Truesdell, *Rational Thermodynamics* (2nd Edn), pp. 116–122, 365–401. Springer-Verlag, New York (1984).
33. B. D. Coleman, M. Fabrizio and D. R. Owen, Il secondo suono nei cristalli: thermodinamica ed equazioni costituive, *Rend. Sem. Mat. Univ. Padova* **58**, 208–227 (1982).
34. M. E. Gurtin, W. O. Williams and W. P. Ziemer, Geometric measurement theory and the axioms of continuum thermodynamics, *Arch. Rational Mech. Anal.* **92**, 1–22 (1986).
35. B. D. Coleman and M. E. Gurtin, Equipresence and constitutive equations for rigid heat conductors, *Z. angew. Math. Physik* **18**, 199–208 (1967).
36. C. Truesdell, *A First Course in Rational Continuum Mechanics* (1st Edn), pp. 37–50. Academic Press, New York (1977).
37. E. C. Young, *Vector and Tensor Analysis* (1st Edn). M. Dekker, New York (1978).